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FIXED POINT RINGS OF FINITE AUTOMORPHISM GROUPS OF HEREDITARY FINITELY GENERATED P.I. ALGEBRAS

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0. Introduction

We consider finite groups acting on hereditary P.I. algebras, which are finite as algebras over their centers. We study the skew group ring and the fixed point ring and try to determine conditions which ensure that these rings again are hereditary.

1. Preliminaries

Let us first suppose that G is a finite group acting on a ring A such that the order of the group $|G|$ is a unit in A. In this case we have the following:

Proposition 1.1. *If A is left hereditary and (A, G, i, α) is a crossed product, then (A, G, i, α) is also left hereditary.*

Moreover the skew group ring and the fixed point are left hereditary if G is a group acting on A.

Proof. We use Maschke's theorem for crossed products [15, Theorem 0.1]. Let I be a left ideal in the crossed product, since the crossed product is a free A-module, I is left A-projective. Now a suitable direct sum of the crossed product maps onto I, hence I has a complement in the direct sum as an A-module, now we are done by Maschke's theorem. Now the result concerning skew group ring is a particular case of the crossed product. Now for the final result let $A*G$ denote the skew group ring; for each idempotent e in $A*G$, $e(A*G)e$ is left hereditary. In particular for $e=1/|G|(\sum_{g\in G} g)$, $e(A*G)e$ is left hereditary, but this ring is isomorphic to the fixed point ring.

This last result is well known [1, Proposition 1.2].

Let us for a moment consider the ordinary group ring $A[G]$ and assume that this

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ring is left hereditary, it is not assumed that the order of the group is a unit in A, but we will show that this in fact is the case. (This result is actually known [6]). Let $x=\sum_{g\in G} g$, then by [14, p. 154, Lemma 2(c)] $l(\omega G)=Ax$, by [17] and the assumpions we get an idempotent e such that $Ax=(A*G)e$, so we get $e=a_0x$ and $x=xe$, thus

$$\sum_{g\in G} g=\left(\sum_{g\in G} x\right)a_0\left(\sum_{g\in G} g\right)$$

a direct computation now shows that $1=a_0|G|$. Thus we have proved:

Proposition 1.2. *Let A be a ring and G a finite group acting on A, then the group ring is left hereditary if and only if A is left hereditary and the order of G is a unit in A.*

In the case of skew group rings the situation is slightly different. First suppose $A*G$ is left hereditary, then so is A and also as before $l(\omega G)=Ax=(A*G)e$, where e is an idempotent in $A*G$. Let $e=a_0x$ and noting that $x=xe$, we get by a straightforward computation that the trace of a_0 $tr(a_0)$ equals 1. Thus a similar argument gives the existence of an element of trace 1 instead of the order of the group being a unit.

Proposition 1.3. *Let A be a ring and G a finite group of automorphisms of A. If the skew group ring is left hereditary, then so is A and the fixed point ring, A^G.*

Proof. We just have to show that A^G is left hereditary, the other result being trivial. The proof depends heavily on the results in [15, Chapter 0]. As we just noted we get an element in A with trace 1, thus the trace gives an A^G epimorphism from A to A^G, thus A is an A^G-generator. We will now use Morita's theorem [15, Theorem 0.4] with $V=A$, $B=A*G$ and $A=A^G$ and we get that A is a finitely generated projective $A*G$-module and $A^G\simeq \mathrm{End}_{A*G}(A)$. In fact A is a cyclic $(A*G)$-module, hence $A\simeq(A*G)e$ for some idempotent e and the isomorphism being an $A*G$ isomorphism. Consequently A^G is isomorphic to $e(A*G)e$, which clearly is left hereditary.

Bergman [2] has shown that for commutative rings the fixed point ring of a finite group of automorphisms of a hereditary ring is hereditary, but clearly $A*G$ need not to be hereditary (suppose G has trivial action on A and apply Proposition 1.2). But the next example shows that the skew group ring might be hereditary without the group ring being hereditary.

Example 1.1. Let g be an outer automorphism of a (commutative) field F, such that the order of g is zero in F. By a theorem of Azumaya and Nakayama [15, Theorem 2.7], $F*G$ is simple artinian, but $F[G]$ is not even hereditary, here G denotes the group generated by g.

The next example shows that the fixed point ring of a finite group of automorphism of a hereditary ring need not to be hereditary.

Example 1.2. Let D be a division ring, $\mathrm{char}(D)=2$ and let g be the inner automorphism of $M_2(D)$ determined by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

it is readily checked that the fixed point ring is isomorphic to $D[x]/x^2$, which is non-hereditary. Finally note that the order of g is zero in D.

In case the order of the group is a regular element in the ring it is also possible to find an example, where the fixed point ring is not hereditary.

Example 1.3 ([15, Example 1.16]). Let A be the integers localized at 2 and consider $M_2(A)$. Let g be the inner automorphism determined by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The fixed point ring is then isomorphic to $A[x]/(x^2-1)$, which is non-hereditary by Proposition 1.2, moreover the order of the group is a regular element in $M_2(A)$.

For more bad examples of fixed point rings of hereditary rings the reader might consult [1, §3].

2. Fixed point rings of hereditary P.I. algebras

In this section A will be a left hereditary p.p.P.I. algebra, which is a finite algebra over its center. In [4, Theorem 4] it is shown that such an algebra is a finite direct sum of a semiprime hereditary module finite algebra and a finite number of algebras each of which has a hereditary noetherian center.

Suppose we are given such an algebra A and an automorphism of A. If

$$A=B\oplus T_1\oplus\cdots\oplus T_k$$

is a decomposition of A, where B is semiprime and T_i has a Dedekind domain as center. Then we claim that the automorphism leaves B and the direct sum of the T_i's invariant. For if e is the identity element of B and f_j the identity element of T_j and we call the automorphism g, then $g(e)$ is a central idempotent and hence either $g(e)$ is in B or there exists a j such that $g(e)f_j=f_j$. Now there must exist a central idempotent x_j, such that $g(x_j)=f_j$, thus $T_j=f_jA=g(x_jeA)$. Thus we have an isomorphism from a semiprime algebra x_jeA to T_j, a contradiction. A similar argument shows that $T_1\oplus\cdots\oplus T_k$ is invariant under g.

Let us also note that each T_i is an upper triangular matrixring of some size, hence an automorphism will take a T_i to a T_j of the same size.

Thus to study fixed point rings for the algebras of this section we can either consider semiprime or non semiprime algebras. For the non semiprime part we have that in case this algebra is twosided hereditary and the order of the group is a non zero element, then the order is a unit, since the center is a field and the results of Section 1 applies.

It is not difficult to give an example of a non semiprime hereditary algebra, such that the fixed point ring is non hereditary.

Example 2.1 (Chatters). Let A be the ring of upper (2×2)-matrices with entries in $Z/2Z$ and let g be defined by

$$g\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & a+b+c \\ 0 & c \end{pmatrix}.$$

g has order 2 and the fixed point ring is isomorphic to $(Z/2Z)[x]/(x)^2$.

Example 2.2. Let B be the ring of Example 1.3 and consider

$$\begin{pmatrix} b & c \\ 0 & q \end{pmatrix}, \quad \text{where } b\in B,\ c\in M_2(Q) \text{ and } q\in Q.$$

B is clearly right hereditary, p.p.P.I. and a finite algebra over its center. If g is the automorphism defined in Example 1.3, then we have an automorphism g_0 of B defined by

$$g_0\begin{pmatrix} b & c \\ 0 & q \end{pmatrix} = \begin{pmatrix} g(b) & g(c) \\ 0 & q \end{pmatrix}.$$

The fixed point ring here is non-hereditary.

For the semiprime case we only have the following rather special result:

Theorem 2.1. *Let G be a finite group of inner automorphisms of a hereditary finitely generated P.I. algebra. If each stalk of A is scalar local, then A^G is hereditary.*

Proof. Since each stalk is scalar local and p.p., each stalk is an integral domain, consequently A is reduced, i.e. has no non zero nilpotent elements and as is well known all idempotents of A are central. It now follows that all idempotents of A are in A^G and moreover A^G contains the center of A. If Z denotes the center of A, we will show that A^G is a finite Z-module and consequently A^G is a finite module over its center. Suppose we have done so, we can argue as follows to complete the proof of theorem:

First A^G is reduced, hence semiprime and we can apply Proposition 2.5 of [6], i.e. we have to show that $(A^G)_x$ is hereditary and the center $Z(A^G)$ is hereditary. Since $B(Z)=B(Z(A^G))$ all rings in question are represented over the same space. It is straightforward to see that $(A_x)^G$ is isomorphic to $(A^G)_x$. A_x is scalar local and hereditary, thus an integral domain. As the maximal ideal is projective it is free, so it must be principal, because A_x is an Ore domain. It now follows that A_x is a serial noetherian domain.

Now suppose that B is a serial noetherian domain and H is a group acting on B. Then the fixed point ring is a serial noetherian domain, because given b_1, b_2 in B^H, there exist x such that $b_1 = xb_2$ (x in B) or there exist y such that $b_2 = yb_1$ (y in B), since B is a domain x (resp. y) must be in B^H. In the same manner one shows that B^H is noetherian in particular B^H is hereditary. We have showed that $(A_x)^G$ is hereditary.

To show that the center of A^G, $Z(A^G)$ is hereditary, we use [2, Theorem 4.4]. The conditions "(d)" and "(b_D)" have been checked, ("(d)" holds since Z is hereditary ([4, Theorem 1, Corollary 2] and $B(Z)=B(Z(A^G)))$. Now if c is regular in $Z(A^G)$, then in each stalk c is a regular element in Z, hence c is regular in Z and "(c)" follows. "(a)" holds, because for an element b in $Z(A^G)$, $r_A(b)=Ae$, but e is a central idempotent, hence $r_{(Z(A^G))}(b)=A^G(e)$.

To show that A^G is a finite Z-module, we will apply Lemma 2.4 of [5], with "$R=Z$, $M=A^G$, $N=A$ and $P=AK$". We can use that lemma, because we have noted that Z is hereditary A is Z-projective by [13, Proposition 1.14] and [10, Theorem 2] and finally AK is K-separable (K denotes the von Neumann regular classical quotient ring of R). Thus by the lemma we just have to show that A^GK is a finitely generated K-module.

AK is a finite K-module [11, Theorem 1.1]. Clearly the action of G extends to an action on AK and the fixed point ring is A^GK.

To make notation easier: We are in following situation, we are given a ring B with center K, such that B is a finite K-module and H is a finite group of inner automorphisms of B. We claim that B^H is a finite K-module.

The action of H on B induced an action H_x on each stalk B_x of B, when we represent B as a ringed space over Spec($B(K)$). Following Jacobson [7, p. 163] we consider Σ the sub-K-algebra of B generated by the units which determine the inner automorphisms in H. Notice that Σ_x is generated by the units, which determine H_x. From [7] we get that

$$\dim_{B_x^{H_x}} B_x = \dim_{K_x} \Sigma_x.$$

Now Σ is a finite K-algebra and Σ_x is for all x a finite K_x-module, thus Σ is a finite K-module. Since Σ is a submodule of B, Σ is K-projective. Thus we can split Spec($B(K)$) in a disjoint unit of open closed sets, such that $\dim_{K_x} \Sigma_x$ is constant for x in each of these sets. Thus without loss of generality we may assume that B_x has constant dimension over $B_x^{H_x}$. Now if $(b_1)_x, \ldots, (b_k)_x$ is a base for B_x over $B_x^{H_x}$, then since B is a finite B^H-algebra of locally constant dimension for all y in some neighbourhood of x, $(b_1)_y, \ldots, (b_k)_y$ will generate B_y and hence is a base. Consequently B is a free B^H-module of rank k, i.e. we have a B^H-isomorphism between B and $\bigoplus B^H$, this isomorphism is also a K-isomorphism, hence B^H is a finite K-module.

The following example shows that extra assumptions are needed for A to be a finite A^G-module, for $|G|$ a regular element and A a hereditary finitely generated P.I. algebra.

Example 2.3. Let A be the ring of all sequences of complex numbers eventually constant. Let G be the group of order two generated by pointwise complex conjugation, clearly A is not a finite A^G-module, but A is commutative hereditary.

The positive result we have got is proved as a combination of known results.

Proposition 2.1. *Let A be a hereditary prime P.I. ring and G a finite group acting on A. Then A is a finite A^G-module.*

Proof. By Robson and Small [16] A is a finite module over its center C and C is a Dedekind domain. By Bergman [1] C^G is Dedekind. Hence by a result of Farkas and Snider [5], C is a finite C^G-module, so A is a finite C^G-module and the proposition is proved.

Proposition 2.2. *Let A be a semiprime p.p.P.I. ring, which is a finite algebra over its center Z and G a finite group acting on A. Then both A and A^G have classical quotient rings and both quotient rings can be obtained by inverting the regular elements of Z^G. Moreover if A has no $|G|$-torsion, then the quotient ring of A^G is von Neumann regular.*

Proof. We will first show that every regular elementof Z^G is regular in Z, hence regular in A (A^G). Let c be regular in Z^G, Z is a p.p. ring [2], so we can write $c=xe$, where x is regular in Z and e is idempotent, e is uniquely determined by c. Since an automorphism takes a regular element to a regular element, e is in Z^G. $(e-1)c=0$, hence $e=1$ and c is regular in Z. Let S denote the regular elements of Z^G and consider the following rings: $B=A_S$, $(A^G)_S$, $C=Z_S$ and $(Z^G)_S$. The elements of G has a natural action on B and C, it is easily seen that $B^G=(A^G)_S$ and $C^G=(Z^G)_S$. Z^G is a p.p. ring [9] and hence C^G is von Neumann regular. If x is a maximal ideal in C^G, then $(C^G)_x$ is a field, where we by the suffix x denote localisation with respect to $C^G\setminus x$, we also consider $(B^G)_x$, $(B)_x$ and $(C)_x$. G induces an action on $(C)_x$ and $(C_x)^G=(C^G)_x$, which is a field hence by a result of Kharchenko [12] C_x is Goldie. Let us also note that if x is in Z, then $\prod_{\sigma\in G}\sigma(x)\in Z^G$, consequently C is the von Neumann regular quotient ring of Z, hence C_x is von Neumann regular. Combining we get that C_x is a finite direct sum of fields. Now C_x is the center of B_x [11, Lemma 1.1] and B_x is semiprime [11, proposition 1.3], thus B_x is semisimple artinian. It now follows that B is the quotient ring of A and that B is von Neumann regular. By [3, Theorem 5.1] $(B^G)_x$ is semiprimary, thus its own quotient ring and therefore B^G is its own quotient ring and the quotient ring of A^G. If A has no $|G|$-torsion, $|G|$ is an invertible element in C, so by Bergmann–Isaac's theorem and a classical theorem of Levitzki $(B^G)_x$ is semisimple artinian and the proof of the proposition is completed.

Notice that Example 1.2 shows that no $|G|$-torsion is essentially for the validity of the last conclusion in Proposition 2.1.

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